

# Mean Field Description of the Fractional Quantum Hall Effect Near $\nu = 1/(2k + 1)$

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## Abstract

The nature of Mean Field Solutions to the Equations of Motion of the Chern–Simons Landau–Ginsberg (CSLG) description of the Fractional Quantum Hall Effect (FQHE) is studied. Beginning with the conventional description of this model at some chemical potential  $\mu_0$  and magnetic field  $B$  corresponding to a “special” filling fraction  $\nu = 2\pi\rho/eB = 1/n$  ( $n = 1, 3, 5 \dots$ ) we show that a deviation of  $\mu$  in a finite range around  $\mu_0$  does not change the Mean Field solution and thus the mean density of particles in the model. This result holds not only for the lowest energy Mean Field solution but for the vortex excitations as well. The vortex configurations do not depend on  $\mu$  in a finite range about  $\mu_0$  in this model. However when  $\mu - \mu_0 < \mu_{cr}^-$  (or  $\mu - \mu_0 > \mu_{cr}^+$ ) the lowest energy Mean Field solution describes a condensate of vortices (or antivortices). We give numerical examples of vortex and antivortex configurations and discuss the range of  $\mu$  and  $\nu$  over which the system of vortices is dilute.

There have been many interesting approaches to understanding the Fractional Quantum Hall Effect (FQHE) in which a two–dimensional electron gas is subjected to a large transverse magnetic field. One such approach which is often called the “Chern–Simons–Landau–Ginzberg” (CSLG) approach to the FQHE [1]– [6] makes use of the fact that in two–dimensions a fermion (such as the electron) can be treated as a boson to which is attached an odd number of magnetic flux quanta. This magnetic flux can be the usual magnetic flux but more conventionally one introduces an additional Gauge field known as the “statistical” gauge field. The electron is then viewed as a boson with an integer number of flux quanta of this “statistical magnetic field”. This “binding” of the statistical flux to the boson is achieved, technically, by the Chern–Simons interaction which is briefly reviewed

below. The key observation in this approach to the FQHE is that the most evident plateaus occur when the (conventional) magnetic flux per particle is an odd integer:

$$\nu = \frac{2\pi\rho}{eB} = \frac{1}{n} \quad n = 1, 3, 5 \dots \quad (1)$$

Thus if we describe each electron as a boson to which is attached an odd number of statistical flux quanta, then the mean statistical magnetic field will cancel the applied magnetic field precisely when Eq. 1 is satisfied. The resulting theory is that of an interacting system of bosons with no net applied field which will Bose condense. The resulting phenomenology has been studied in detail in Refs. [1]- [6].

There has also been considerable work on describing the Fractional Quantum Hall system when the density (or the magnetic field) is perturbed away from the special filling fraction described by Eq. 1. In the CSLG description vortices occur in the system to accommodate these fluctuations in the density or in the magnetic field. It has been shown that such vortices **do** exist and they can be found by perturbing the ground state of the system at the special filling fraction by adding (or removing) particles to the system. In this way single vortex solutions can be found [1], [5]. These vortices are in fact the Laughlin quasi-particles and quasi-holes [7]. When they occur in pairs they are neutral excitations analogous to the rotons of 3-D superfluids.

Vortices also form when the external magnetic field is changed with the number of particles kept fixed. In this case the perturbation is a constant, uniform shift of the field rather than a local density perturbation. Thus, in the limit of infinite volume, a lattice of vortices covering the entire space is expected to form just as would happen if the mean particle density would be perturbed from that of Eq. (1). The conventional way to analyze this system is to begin at the special filling fraction (i.e. to consider first the case in which the density has been modified so that  $2\pi\rho/eB = 1/n$ ) and to accommodate the excess density by means of these vortex excitations. In this paper we shall take a slightly different approach. The electron density is described by means of a chemical potential  $\mu$ . When the magnetic field is changed away from the special filling fraction we shall try to find the Mean Field ground state of the resulting system by looking for a solution to the resulting equations of motion. Since, in this case, the mean Statistical Field does not cancel the applied field there will be no uniform mean field solution and the Mean Field equations are extremely “frustrated”. We shall discuss the properties of solutions to this system of equations analytically and then present some numerical solutions.

Our main goal in this paper is to display some novel features of the CSLG description of the FQH system. To this end we adopt the simplified version of the model which is described in Ref. [4] which allows us to show these features more clearly. We begin with a gas of bosons represented by a (nonrelativistic) scalar field  $\psi$  with a mass  $m$ . The fact that these bosons are at some finite density  $\rho$  will be implemented by considering the system at some chemical potential  $\mu$ . The system will also have a Statistical Gauge Field  $a_\mu$  whose main purpose is to attach an integer number of statistical magnetic flux quanta to each boson thus allowing them to describe an electron gas. Our main simplification relative to the actual physical situation is to imagine that these bosons have a hard core repulsive interaction ( $\propto |\psi|^4$ ) instead of a Coulomb interaction. These bosons are also subjected to a fixed external magnetic field  $B$  which is described by an electromagnetic potential  $A_\mu$ . The Lagrangian for this system is given (in units for which  $\hbar = c = 1$ ) by:

$$\begin{aligned}\mathcal{L} = & -i\psi^\dagger(\partial_0 - i(eA_0 - ga_0)\psi + \frac{1}{2m}|\nabla - i(e\mathbf{A} - g\mathbf{a})\psi|^2 \\ & - \frac{g^2}{4\pi n}\varepsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda + \frac{\lambda}{2}(\psi^\dagger\psi)^2 - \mu\psi^\dagger\psi\end{aligned}\quad (2)$$

We shall see in a moment that in order for  $\psi$  to describe electrons  $n$  must be an odd integer. The third term in this Lagrangian is the Chern-Simons term which is responsible for the fact that Statistical Magnetic Flux is attached to each particle. The equation of motion for the field  $a_0$ ,

$$\frac{g}{2\pi n}b = \psi^\dagger\psi \quad (3)$$

implies that to each particle (with  $\int d^2x\psi^\dagger\psi = 1$ ) is attached a Statistical magnetic flux  $g\Phi = 2\pi n$ . Thus if  $n$  is an odd integer each boson has an odd number of Statistical flux quanta attached to it so that it has Fermi statistics [11].

In the special case when the filling fraction

$$\nu = \frac{2\pi\rho}{eB} = \frac{1}{n} \quad (4)$$

(where  $\rho = \psi^\dagger\psi$  is the density of particles) the average effect of the Statistical field precisely cancels that of the externally applied magnetic field i.e.  $g < b > = eB$  so that a gauge can be chosen in which, on the average,  $g < a_i > = eA_i$ . This, of course, corresponds to the special filling fraction and leads to the CSLG description of the Fractional Quantum Hall Effect.

We begin our discussion by writing down the equations of motion which result from varying  $\psi$ ,  $a_0$  and  $a_i$  in the Lagrangian (2). Since  $A_\mu$  is the externally applied Gauge Potential due to a spatially constant magnetic field we choose  $A_0 = 0$ .

$$i(\partial_0 + iga_0)\psi + \frac{D_i^2}{2m}\psi - \lambda(\psi^\dagger\psi)\psi + \mu\psi = 0 \quad (5)$$

$$\psi^\dagger\psi = \frac{gb}{2\pi n} \quad (6)$$

$$\frac{i}{2m}(\psi^\dagger D_i\psi - (D_i\psi)^\dagger\psi) = \frac{g}{2\pi n}\varepsilon_{ij}(-\partial_0 a_j + \partial_j a_0) \equiv \frac{g}{2\pi n}\varepsilon_{ij}e^j \quad (7)$$

where  $D_i = \partial_i - ieA_i + iga_i$  is the covariant derivative and  $e_j$  is the statistical electric field. The first equation (5) is the generalized (nonlinear) Shroedinger equation for this system. The second equation (6) implements the constraint that there are  $n$  flux quanta per particle as discussed above. The third equation which also results from the Chern-Simons term in the action relates the statistical electric field to the particle current.

The above equations of motion are **operator** equations for the Quantum Fields  $\psi$  and  $a_i$ . The technique which we use to study these Quantum Equations (see Ref. [1]- [6]) is to first find a Mean Field Solution to the above equations. The properties of the Quantum

system can then be studied by expanding about this mean field. We thus begin by solving the equations of motion (5-7) as if they were classical equations. When

$$\frac{\mu}{\lambda} = \frac{eB}{2\pi n} \quad (8)$$

this set of equations has the constant solution  $|\psi|^2 = \frac{\mu}{\lambda}$ ,  $e_j = 0$ , and  $gb = eB$ . If either  $\mu$  or  $B$  is changed so that Eq. (8) is not satisfied it would seem difficult to find solutions which simultaneously minimize the potential and produce no net field. We now analyze the equations in this case in a more systematic manner.

We begin with several important observations about these equations which are generalizations of theorems proven in Ref [12] in a somewhat different context. The first observation is that unless  $\mu/\lambda = eB/(2\pi n)$  (which corresponds to the special filling fraction) there are no nonzero covariantly constant solutions (solutions with  $D_\mu\psi = 0$ ) to this system of equations. The proof is as follows: first write  $\psi = \xi e^{i\Omega}$ . Now  $D_\mu\psi = 0$  implies that  $\partial_\mu\xi = 0$  and that  $\partial_\mu\Omega = eA_\mu - ga_\mu$  so that the combination  $eA - ga$  is a pure gauge. It follows that  $eB = gb$ . Now Eqs. (5) and (6) with  $D_\mu\psi = 0$  imply that either  $\psi = 0$  or that  $\mu/\lambda = eb/(2\pi n)$  so that  $\mu/\lambda = eB/(2\pi n)$ . This completes the proof that nonzero covariantly constant solutions only exist when  $\mu/\lambda = eB/(2\pi n)$ .

In addition to the lack of solutions with  $D_\mu\psi = 0$  there are also no nonzero covariantly static solutions (i.e. solutions with  $D_0\psi = 0$ ) unless  $\mu/\lambda = eB/(2\pi n)$ . The proof again begins by writing  $\psi = \xi e^{i\Omega}$ .  $D_0\psi = 0$  then implies that  $\partial_0\xi = 0$  and that  $\partial_0\Omega = eA_0 - ga_0 = -ga_0$ . Using this and the third equation of motion Eq. (7) as well as the fact that  $\partial_0A_i = 0$  we find that

$$\partial_0(\partial_j\Omega + ga_j - eA_j) \propto \epsilon_{jk}(\partial_k\Omega + ga_k - eA_k)\xi^2 \quad (9)$$

Since  $\xi$  is independent of time we can identify three possibilities. The first option is  $\xi = 0$  in which case  $\psi = 0$ . A second possibility is that  $\partial_j\Omega + ga_j - eA_j = 0$  in which case  $gb = eB$ . In this case Eq. (6) (the Chern-Simons constraint) implies that  $|\psi|^2 = eB/2\pi n$  which is the condition for the special filling fraction. This further implies that  $\xi$  is spatially constant which, from Eq. (5) forces  $|\psi|^2 = \mu/\lambda$ . The third possibility for satisfying Eq. (9), which is necessary when we have a nonzero  $\psi$  with  $\mu/\lambda \neq eB/(2\pi n)$  is that  $\partial_j\Omega + ga_j - eA_j$  is a nonzero oscillating (sinusoidal) function of time. It follows that  $gb - eB$  and thus  $gb$  is an oscillating function of time which then implies that  $\xi$  oscillates with time which contradicts the equation  $\partial_0\xi = 0$ . It follows that except at special filling fractions there is no nonzero solution which is covariantly static.

Despite the fact that no covariantly static solution exists it *is* possible to find nonzero time independent solutions (with  $\partial_0\psi = 0$ ) but with  $a_0 \neq 0$ . As is shown (in a somewhat different context) in Ref. ([12]) this can be done by extremizing the energy functional

$$H = \int d^2x \mathcal{H} = \int d^2x \left[ \frac{1}{2m} |D_i\psi|^2 - \mu|\psi|^2 + \frac{\lambda}{2} |\psi|^4 \right] \quad (10)$$

with  $D_i = \partial_i - ieA_i + iga_i$  but subject to to the constraint Eq. (6) that

$$\nabla \times a = \frac{2\pi n}{g} |\psi|^2 \quad (11)$$

In other words when Eq. (10) is varied with respect to  $\psi$  and  $a_i$  subject to the constraint (11) one obtains precisely the equations of motion (5–7) with  $\partial_0\psi = 0$  but with  $a_0 \neq 0$ . The term proportional to  $a_0$  appears when  $a_i$  is varied with respect to  $\psi$  as required by the constraint Eq. (11).  $a_0$  will be a solution to Eq. (7) with  $\partial_0 a_i = 0$  (which can always be arranged since  $\psi$  is time independent).

Our job is now to find a configuration of  $\psi$  and  $a_i$  which minimizes the energy (10) subject to the constraint (11) for various values of the parameters  $\mu$  and  $B$ . The simplest case which has already been discussed is the case in which  $\mu/\lambda = eB/(2\pi n)$ . In this case the configuration with  $\rho = \mu/\lambda$  both minimizes the potential and forces  $eb = gB$  which admits a configuration with  $D_i\psi = 0$ . In this situation the density  $\rho$  satisfies  $\nu = 2\pi\rho/eB = 1/n$  which is the special filling fraction. The case of interest to us however occurs when either  $B$  or  $\mu$  is modified so that

$$\frac{eB}{2\pi n} = f\frac{\mu}{\lambda} \neq \frac{\mu}{\lambda} \quad (12)$$

In this case there is a problem. If we force the potential to be at its minimum  $|\psi|^2 = \mu/\lambda$  then  $eB \neq gb$  and the  $|D_i\psi|^2$  term in the Hamiltonian (10) leads to an infinite energy (the Meissner effect). If, on the other hand, we force  $eB = gb$  (which must then be spatially constant) then  $|\psi|^2 = f\mu/\lambda$  which is not at the minimum of the potential.

The solution to the above problem is as follows: Suppose we first consider the configuration described above in which  $|\psi|^2 = f\mu/\lambda$  is *not* at the minimum of the potential. It may be surprising, but the fact is that this configuration *is* an extremum of the Hamiltonian (10) subject to the constraint (11). To see this notice that this configuration *does* satisfy the equations of motion (5–7) but with

$$ga_0 = \frac{\mu^2}{\lambda} f(1-f) \quad (13)$$

which is spatially constant. We see from Eq. (5) that the constant part of  $a_0$  behaves in the same manner as a shift in the chemical potential thus allowing the minimum of the potential to be consistent with the cancellation of  $eB$  and  $gb$ .

It is also clear intuitively why this configuration is an extremum of the Hamiltonian (10). The main candidate for a fluctuation which might lower the energy is one in which  $\psi$  is changed locally by an infinitesimal amount in the direction of the minimum of the potential. This will cause an infinitesimal amount of flux ( $f(eB - gb)$ ) to thread through the system. If this flux is truly infinitesimal it will be less than one flux quantum and thus its gauge potential will lead to an infinite contribution to the gradient term in the Hamiltonian. (It is of course possible to consider fluctuations which have zero total flux but there is no reason to believe that these would lower the energy. The fact that they leave the energy unchanged to lowest order is evidenced by the previous argument that this configuration solves the equations of motion.)

The configurations in which the flux is not infinitesimal but rather consists of an integral number of flux quanta *is* a finite energy excitation of this system. If, for example, we imagine modifying the chemical potential away from the special value we might hope that these vortices would lower the energy of the system and thus describe the Fractional Quantum Hall system *away* from special filling as a collection of these vortices. Unfortunately this

is not the case. We shall see shortly that for small deviations of  $\mu$  from its “special” value these vortices always *increase* the energy. Let us first discuss why this is the case and then explain its consequences.

The main point is that the configurations which are extrema of the Hamiltonian (10) subject to the constraint (11) are completely independent of the value of the chemical potential  $\mu$ ! We have, in fact, proven this already. Configurations are extrema of (10) subject to (11) if and only if they satisfy the equations of motion (5–7). Thus if a particular  $\psi$  is such an extremum (it may be a constant or, more generally, a multivortex configuration) with a given value  $\mu_1$  of  $\mu$  then it will satisfy the equations of motion for some function  $a_0^{(1)}$ . The same  $\psi$  will satisfy the equations of motion for any other value  $\mu_2$  of  $\mu$  but this time with a new  $a_0^{(2)} = a_0^{(1)} + \mu_2 - \mu_1$ . It is thus also an extremum of the Hamiltonian with this new value  $\mu_2$  of  $\mu$ .

Even though the extremal configurations for differing values of  $\mu$  are the same, the energetics may differ for different values of  $\mu$ . Notice, for example, that for  $\mu$  such that  $\mu/\lambda = eB/(2\pi n)$  (let us call this value of the chemical potential  $\mu_0$ ), the configuration  $|\psi|^2 = \mu_0/\lambda$  clearly has the lowest possible energy. A single vortex configuration  $\psi_v(x)$  which solves the equations of motion (see [13]) will have a larger energy than the ground state. Let  $\epsilon_v$  be the excess energy of the vortex with respect to the ground state of the Hamiltonian with  $\mu = \mu_0$ . Now consider an alternate Hamiltonian with  $\mu = \mu_1 \neq \mu_0$ . The same configuration  $\psi_v(x)$  will still be an extremum of this Hamiltonian but its energy  $\epsilon_v^{(1)}$  (which is the *difference* in energy between the vortex configuration and the configuration with  $|\psi|^2 = \mu_0/\lambda$ ) will differ:

$$\epsilon_v^{(1)} = \epsilon_v - (\mu_1 - \mu_0) \int d^2x \left( |\psi_v(x)|^2 - \frac{\mu_0}{\lambda} \right) = \epsilon_v - (\mu_1 - \mu_0) N_v \quad (14)$$

where  $N_v = \pm 1/n$  is the “particle number” of the vortex. Note that when  $\mu$  is decreased it is preferable to form a vortex ( $N_v < 0$ ) whereas if  $\mu$  is increased an antivortex is preferred.

Equation (14) has the following consequences. **For small values** of  $\delta\mu = \mu_0 - \mu_1$  the vortex configuration *increases* the energy of the system at  $\mu = \mu_1$ . We thus expect that the constant configuration with  $|\psi|^2 = \mu_0/\lambda$  will be the configuration of lowest energy despite the fact that the potential energy is not at its minimum. This configuration has a density  $\rho = \mu_0/\lambda$  so that the system remains at the special filling fraction even after  $\mu$  has been shifted from  $\mu_0$  to  $\mu_1$ . (We emphasize again that this occurs for small shifts  $\mu_0 - \mu_1$ .) It follows that in the Mean Field approximation

$$\left. \frac{d\rho}{d\mu} \right|_B = 0 \quad (15)$$

for a range of  $\mu$  near  $\mu = \mu_0$ . This equation is familiar from the integer Quantum Hall effect and is due to the presence of a gap in the spectrum. (See Refs. [14] for a discussion of this with respect to the FQHE). It implies that a finite change in the chemical potential is required before the density can be modified. Returning to Eq. (14) we see that when  $\delta\mu = |\mu_0 - \mu_1| \geq \epsilon_v/N_v$  a single vortex has *lower* energy than the constant configuration  $|\psi|^2 = \mu_0/\lambda$ . A weakly interacting gas of such vortices will have an even lower energy. We thus expect that near this value ( $\epsilon_v/N_v$ ) of  $\delta\mu$  the lowest energy configuration of the

Hamiltonian (10) subject to the constraint (11) will be a collection of vortices (which will likely form a lattice in the Mean Field approximation).

The situation is similar when the magnetic field is modified instead of  $\mu$ . If we begin at the special filling fraction with  $eB = 2\pi n\mu_0/\lambda$  and change  $B$  at fixed  $\mu$  by a small amount, the lowest energy configuration of the Hamiltonian will occur at a *new* density for which  $2\pi\rho/eB$  is still equal to  $1/n$  but which will now *not* be at the minimum of the potential. As the magnetic field is increased (or decreased) further (again at fixed  $\mu = \mu_0$ ) the cost in energy of a single vortex (or antivortex) becomes progressively smaller until at some critical value of the field it becomes negative. At that point the lowest energy mean field configuration is no longer a constant but rather a lattice of vortices in which case  $2\pi\rho/eB$  is no longer equal to  $1/n$ . (If the magnetic field is increased then the vortices will “condense” whereas if it is decreased the antivortices will “condense”.) Thus if  $2\pi\rho/eB$  is plotted either as a function of  $\mu$  or as a function of  $B$  there is a plateau surrounding the value  $\mu_0$  for which this ratio is constant and equal to  $1/n$ . If, on the other hand, the  $B$  is varied at fixed *density* then we move off the plateau and the lowest energy mean field configuration consists of a vortex lattice.

In the remainder of this paper we shall look more closely at the vortices of this model which, as we have discussed, will be solutions both for  $\mu = \mu_0$  and for values of  $\mu$  differing from  $\mu_0$ . We shall present some numerical solutions for these vortices which will allow us to estimate the value of  $\mu$  at which a lattice of vortices starts to form. The shape of the vortices will also lead to an estimate of the density at which the collection of vortices becomes non-dilute. The method we have chosen for finding vortices is by considering the Hamiltonian and constraint given by Eqs. (10–11) at a value of  $\mu = \mu_0 = eB\lambda/(2\pi n)$ . We then look for radially symmetric configurations (which necessarily carry an integer number of flux quanta of  $eB - gb$ ) which minimize the Hamiltonian.

Anticipating the fact that our solution will be a vortex with an integer number of flux quanta we organize a radial ansatz as follows: First write

$$\psi(r, \theta) = \xi(r)e^{-ik\theta} \quad (16)$$

with  $k$  an integer. The Hamiltonian (10) can now be written as:

$$H = \int_0^\infty 2\pi r dr \left[ \frac{1}{2m} \left\{ \left( \frac{d\xi(r)}{dr} \right)^2 + \left( \frac{k}{r} + \frac{eBr}{2} - \frac{g}{r} \int_0^r \hat{r} b(\hat{r}) d\hat{r} \right)^2 \xi^2(r) \right\} - \mu \xi^2(r) + \frac{\lambda}{2} \xi^4(r) \right] \quad (17)$$

with the constraint

$$\xi^2(r) = \frac{gb(r)}{2\pi n} \quad (18)$$

(Note that if we were considering a value of  $\mu$  not equal to  $\mu_0$  we would still use the above equation but with  $\mu = \mu_0/f$  so that the Mean Field Solution would be  $\xi^2 = f\mu/\lambda$ .) We now define the function  $h(r)$  via the formula

$$gb(r) - eB = \frac{h'(r)}{r} \quad (19)$$

where  $h'(r) = dh/dr$ .  $h(0)$  can be chosen equal to 0 without loss of generality. The second term in Eq. (17) is then proportional to:

$$\left(\frac{k}{r} + \frac{eBr}{2} - \frac{g}{r} \int_0^r r' b(r') dr'\right)^2 \xi^2(r) = \left(\frac{k}{r} - \frac{h(r)}{r}\right)^2 \xi^2(r) \quad (20)$$

In order for the integral to be finite at large  $r$  we require  $h(\infty) = k$  (or, more precisely, we require it to be an integer and we choose  $k$  in Eq. (16) to be that integer). (We only consider the cases  $k = \pm 1$  in this paper, since those configurations have the lowest energy.) Furthermore, as is standard for all vortices,  $\xi^2(r)$  must vanish at the origin in order for the energy to be finite. The Chern–Simons condition Eq. (18) then implies

$$\xi^2(r) = \frac{gb}{2\pi n} = \frac{eB + h'(r)/r}{2\pi n} \geq 0 \quad \text{and} \quad \rightarrow 0 \quad \text{as} \quad r \rightarrow 0 \quad (21)$$

This is the most difficult condition to implement in a numerical scheme in which the function  $h(r)$  is varied to minimize the energy.

With the above definitions the Hamiltonian is given by

$$H = 2\pi \int_0^\infty r dr \left[ \frac{1}{2m} \left( \frac{d\xi(r)}{dr} \right)^2 + \frac{1}{2m} \left( \frac{k}{r} - \frac{h(r)}{r} \right)^2 \xi^2(r) - \mu \xi^2(r) + \frac{\lambda}{2} \xi^4(r) \right] \quad (22)$$

with  $h(r)$  chosen so that  $h(0) = 0$ ,  $h(\infty) = k$  and  $\xi^2(r)$ , defined by Eq. (21), is  $\geq 0$ . Notice that these conditions guarantee that the total flux of the vortex

$$\int d^2x (gb - eB) = 2\pi k \quad (23)$$

The final step is to subtract, from the energy of the vortex solution of Eq. (22), the energy of the Mean Field solution  $\xi^2 = f\mu/\lambda$ . This results in a vortex energy given by:

$$H_v = 2\pi \int_0^\infty r dr \left[ \frac{1}{2m} \left( \frac{d\xi(r)}{dr} \right)^2 + \frac{1}{2m} \left( \frac{k}{r} - \frac{h(r)}{r} \right)^2 \xi^2(r) - \mu(1-f) \left( \xi^2(r) - f \frac{\mu}{\lambda} \right) + \frac{\lambda}{2} \left( \xi^2(r) - f \frac{\mu}{\lambda} \right)^2 \right] \quad (24)$$

The procedure at this stage is to search, numerically, through the space of such functions  $h(r)$  until the Hamiltonian is minimized.

One point which is clear is that the form of the vortex solution (for which  $f(eB - gb) > 0$ ) is quite different from that of the antivortex solution. The reason is that  $\xi(r)$  and thus  $b(r)$  must vanish at the origin. As a consequence the density  $\psi^2(r) \propto b(r)$  for the vortex solution can be a monotonic function of  $r$  which increases from 0 at the origin and reached  $eB$  at infinity. The antivortex solution must however be zero at the origin then increase to a value greater than  $eB$  (so that  $f(eB - gb) < 0$ ) and then decrease again to attain its asymptotic value  $eB$  as  $r \rightarrow \infty$ . We shall of course see this behavior clearly in the numerical solutions shown in Figures 1-4 below.

For the numerical work we chose some representative values for the parameters of the model [9]:



$$\begin{aligned}
\mu_0 &= .010 \text{ eV} \\
\mu_0/\lambda &= 10^3 \mu\text{m}^{-2} \\
m &= .08 m_e
\end{aligned}
\tag{25}$$

(In the units  $\hbar = c = 1$  this translates to  $\lambda = .00025 \text{ eV}^{-1}$ ,  $\rho = \mu_0/\lambda = 40 \text{ eV}^2$ ,  $eB = 250n \text{ eV}^2$  and  $m = 41000 \text{ eV}$ .) Note that  $\mu_0/\lambda$  is the density of carriers. (Although the above values were chosen to be representative of experiments which exhibit the FQHE it is difficult to call them “realistic” since the CSLG model has been greatly simplified by replacing the Coulomb interaction of the charge carriers with a short range interaction.)

In Fig. 1 and Fig. 2 we present the numerical solution for the **vortex** configurations (in which  $gb$  is lowered relative to  $eB$  and  $h(r) \rightarrow -1$  as  $r \rightarrow \infty$ ). We plot the functions  $h(r)$  and the density  $\rho(r)$  for  $\nu = 1/3, 1/5$  and  $1/7$  respectively. When changing  $\nu$ , the density  $\rho$  remains fixed as the magnetic field  $B$  is varied. Figures 3 and 4 contain plots of  $h(r)$  and  $\rho(r)$  for the same values of  $\nu$  but now for the **antivortex** configurations. In Table I we present the energies and some measure  $r_0$  of the size of each vortex and antivortex. We have arbitrarily chosen the size of the vortex as the value of  $r$  at which the energy density has reached 99% of its total value.

We are now ready to describe quantitatively (within this model) what happens when the chemical potential is varied from  $\mu_0$ . As discussed in great detail in this paper there is no change in the density unless  $\mu - \mu_0$  is approximately equal to the energy of a vortex times  $n$  (i.e. the energy per particle of the vortex). We can now see, quantitatively, how this works from Eq. (24). If  $f \neq 1$  so that  $\mu = \mu_0/f \neq \mu_0$  then the energy of the vortex is simply

$$\epsilon_v(\mu) = \epsilon_v(\mu_0) - (\mu - \mu_0) \times \left( \mp \frac{1}{n} \right) \tag{26}$$

where the minus sign is for a vortex and the plus sign for an antivortex. Thus for  $\mu < \mu_0$  the vortex configuration has lower energy than the antivortex configuration. We naively expect that when  $\mu_0 - \mu = n\epsilon_v(\mu_0)$  (or near this point) the Mean Field ground state should be a condensate (possibly a lattice) of vortices. Conversely when  $\mu > \mu_0$  the antivortex has lower energy and when  $\mu - \mu_0 = n\epsilon_v(\mu_0)$  we naively expect a condensate of antivortices. Unfortunately, for our choice of parameters,  $\epsilon_v$  is quite large. Thus the value  $\mu_{cr}$  or  $\mu$  at which this condensate occurs in the above naive calculation and which is shown in Table I differs from  $\mu_0$  by an unreasonably large amount. This leads, in particular, to a negative value of  $\mu_{cr}$  for the  $n = 5$  and  $n = 7$  vortex configurations. In fact, as  $\mu$  is varied from  $\mu_0$  towards  $\mu_{cr}$ , another critical value  $\hat{\mu}$  of  $\mu$  is reached at which  $2\pi\hat{\mu}/\lambda eB = 1/(n+2)$  well before  $\mu_{cr}$  is reached. At  $\hat{\mu}$  the system is better described by a Chern–Simons theory with the new value  $\hat{\nu} = 1/(n+2)$  of the filling fraction.

In light of the above remarks we should try to understand whether in fact one does form a vortex condensate in our model at our chosen values of the parameters. Certainly the vortices must be strongly interacting (or overlapping) well before  $\hat{\mu}$  or  $\mu_{cr}$  is reached. This can be better understood by first supposing that such a condensate is formed as  $\mu$  is lowered. The approximate density  $\rho_1$  at which a description of this condensate in terms of the single vortex solutions presented above fails depends most prominently on the size of a vortex. In Table II we show the approximate density of vortices  $\rho_{v1}$  and chemical potential  $\mu_1$  at which the vortices begin to “touch”. For a hexagonal lattice of vortices this will occur when

$\rho_{v1} = 1/(2\sqrt{3}d^2)$ , ie, where  $d$  (the distance between vortices) equals the size of the vortices. Table II also shows the corresponding values of the filling fraction. Notice that in most cases the vortex condensate becomes dense well before the “next” value of  $n$  (i.e. well before  $2\pi\rho/eB = 1/(n \pm 2)$ ). We conclude from this that an approximation in terms of a dilute gas of vortices breaks down well before  $\mu = \mu_{cr}$ . It is thus likely that even for our chosen values of the parameters a condensate of vortices will form in the Mean Field description. The formation of this vortex condensate and the resultant pinning of the vortices is what gives rise to the hall plateaus by allowing the system to continue to behave as if it were in a state  $\nu = 1/n$  even after the magnetic field or the chemical potential has been changed to move it away from that value. This is because the (anti)vortices accommodate the localized excess (deficit) of charge.

### Summary

In this paper we have studied the Mean Field behavior of the CSLG description of the FQHE when the filling fraction deviates from the “special” filling fraction for which  $\nu = 1/n$  with  $n$  an odd integer. We have shown how the Field Theoretic description of this model at a fixed chemical potential  $\mu$  and magnetic field  $B$  can be studied for a range of  $\mu$  surrounding the value  $\mu_0$  corresponding to the special filling fraction. For small values of  $|\mu - \mu_0|$  (and at zero temperature) the density is independent of  $\mu$ . This is reminiscent of what occurs for the integer Quantum Hall Effect. As  $\mu$  is decreased beyond some  $\mu_{cr}^-$  we show how the homogeneous Mean Field configuration is unstable to the formation of a condensate of vortices. If  $\mu$  is increased above some  $\mu_{cr}^+$  the instability is to the formation of a condensate of antivortices. We have presented a numerical example of these vortex and antivortex configurations and we estimated the densities and filling fractions at which the description in terms of a noninteracting system of vortices breaks down.

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# TABLES

vortex/antivortex	$n$	size ( $\mu m$ )	energy ( $\times 10^{-2} eV$ )	$\mu_{cr}$ ( $\times 10^{-2} eV$ )
vortex	3	0.027	0.25	0.25
	5	0.018	0.21	-0.05
	7	0.015	0.20	-0.4
antivortex	3	0.035	0.83	3.5
	5	0.025	0.80	5.0
	7	0.020	0.79	6.5

TABLE I. Energy and size ( $r_0$ ) of vortices and antivortices for  $n = 3, 5$  and  $7$ .  $\mu_{cr}$  is a naive estimate of the value of the chemical potential at which a condensate of these configurations is expected to form ( $\mu_0 = 10^{-2} eV$ ).  $\mu_{cr}$  is more carefully described in the text.

vortex/antivortex	$n$	$\rho_{v1}$ ( $\mu m^{-2}$ )	$\nu_1^{-1}$
vortex	3	400	2.6
	5	890	4.1
	7	1280	5.7
antivortex	3	240	3.2
	5	460	5.5
	7	720	7.7

TABLE II. The density  $\rho_{v1}$  and the corresponding filling fraction  $\nu_1$  at which the condensate of vortices is expected to become dense. (See text for a precise definition.)

# FIGURES

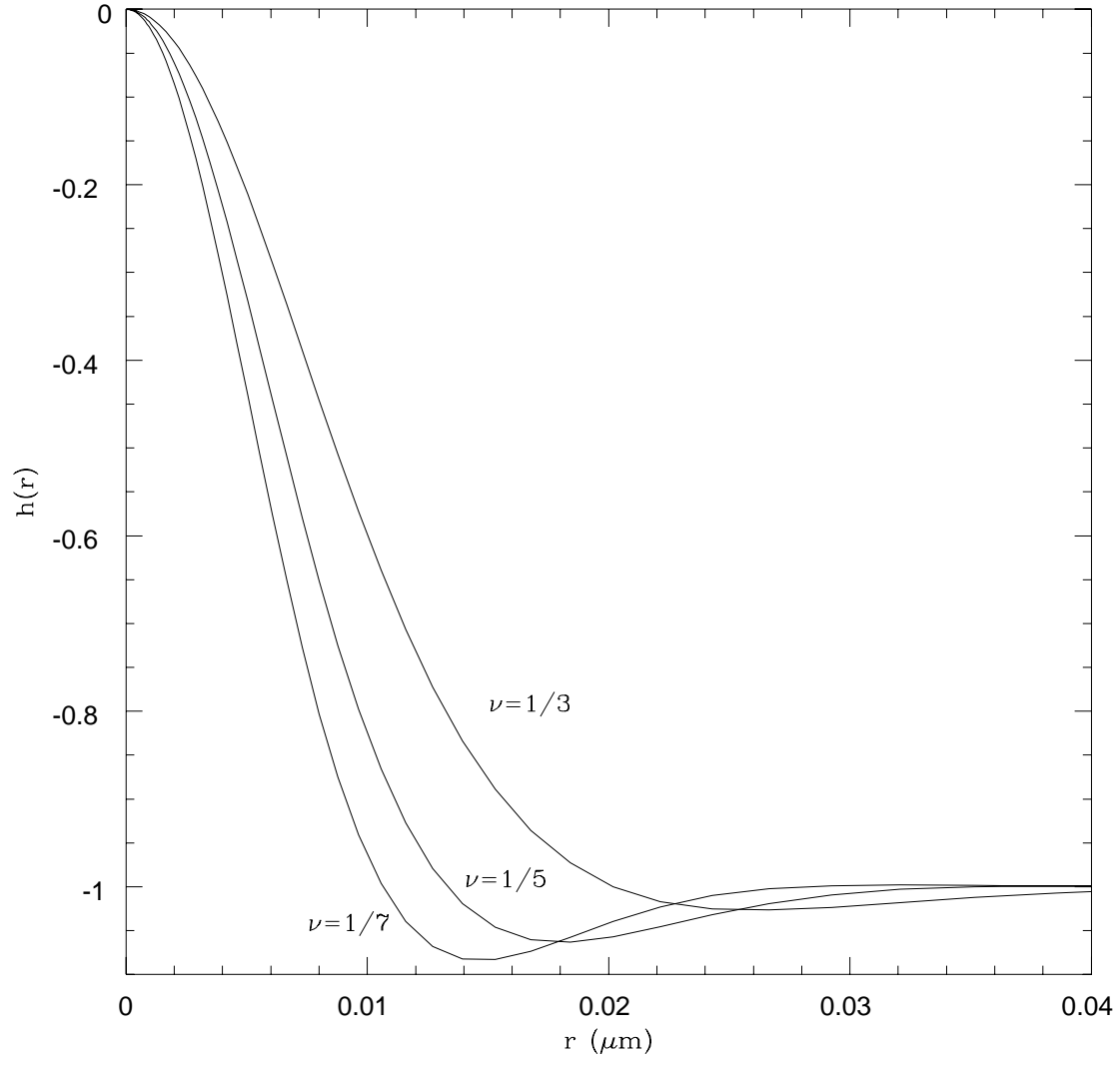


FIG. 1. The function  $h(r)$  corresponding to a vortex for  $\nu = 1/3, 1/5$  and  $1/7$ .

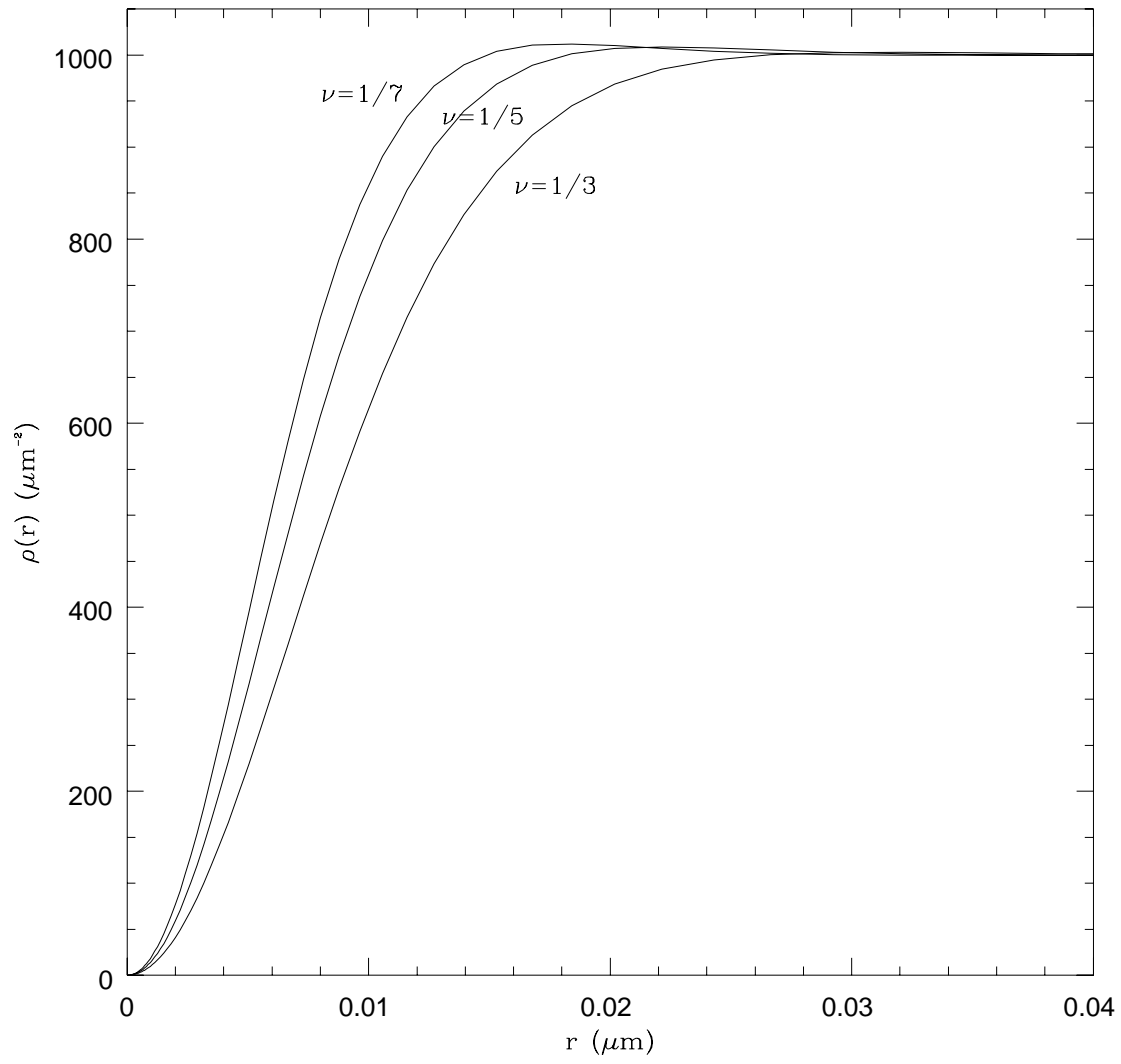


FIG. 2. Density profile of the vortex configuration for  $\nu = 1/3, 1/5$  and  $1/7$ .

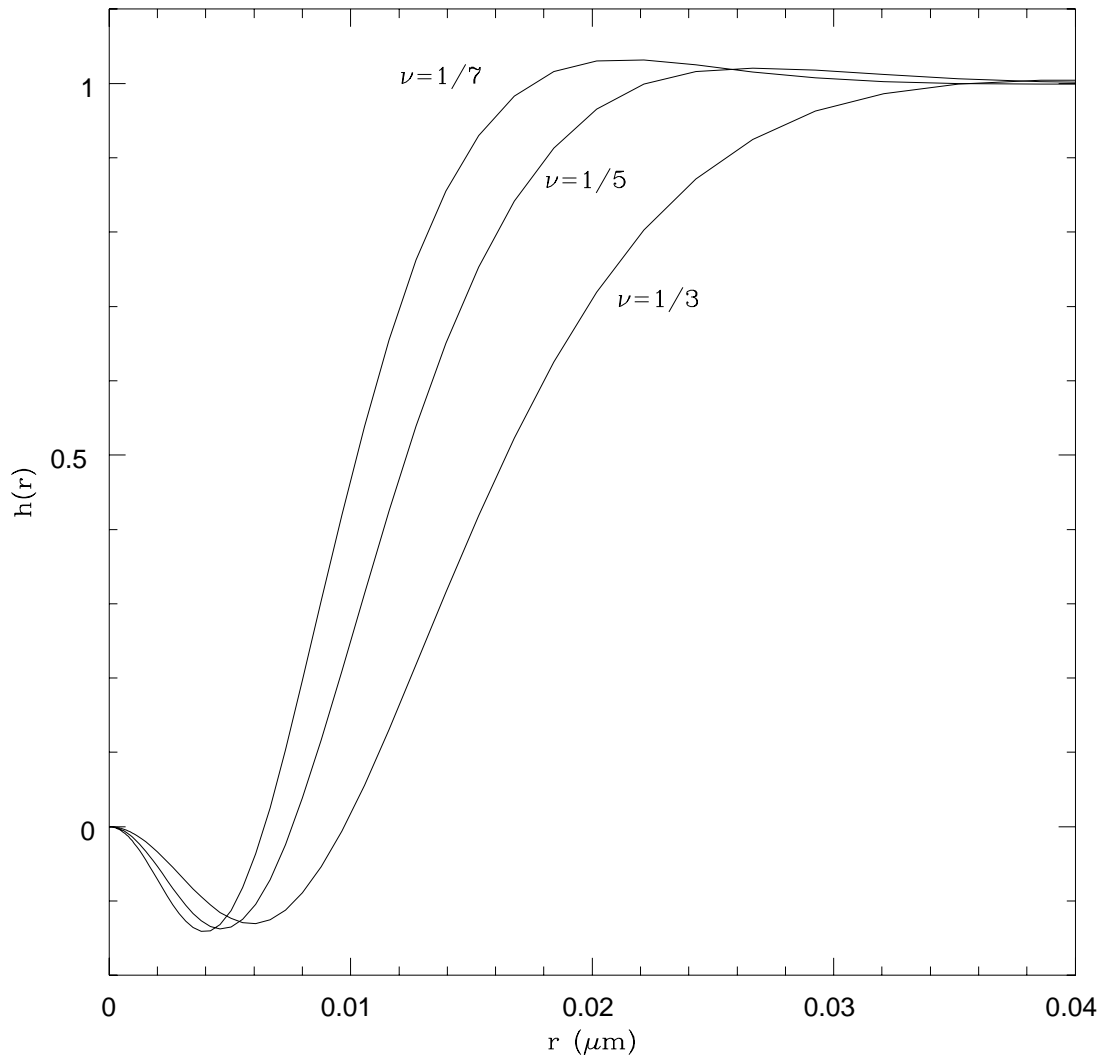


FIG. 3. The function  $h(r)$  corresponding to an antivortex for  $\nu = 1/3, 1/5$  and  $1/7$ .

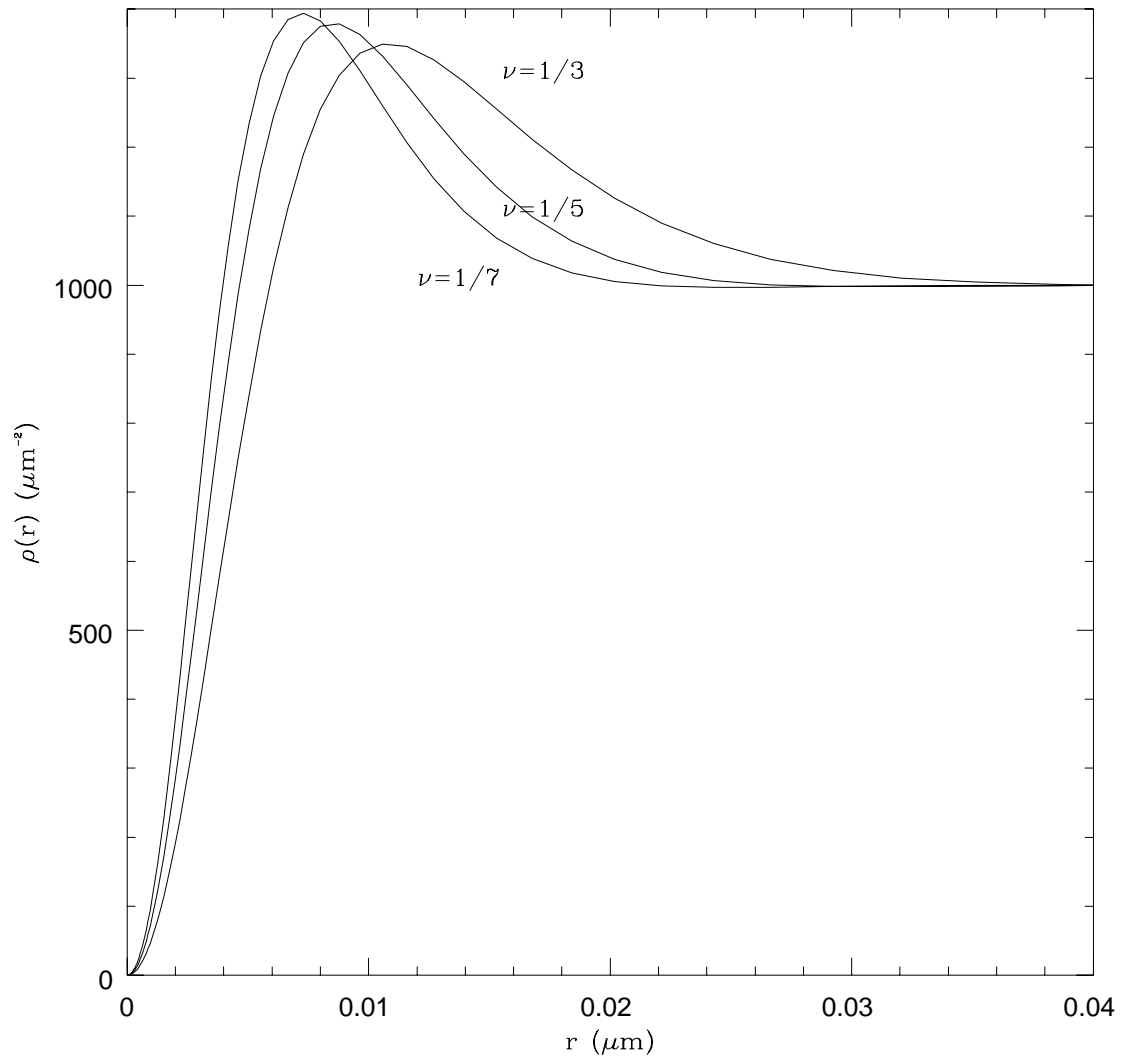


FIG. 4. Density profile of the antivortex configuration for  $\nu = 1/3, 1/5$  and  $1/7$ .